

Level correlations in disordered superconducting grains

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I study the quasiparticle level correlations in a grain of a weakly disordered d-wave superconductor, and show that, in a wide intermediate energy range, they are characterized by a novel type of universal behavior.

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Level correlations represent a fundamental property of an electron system. They reflect sensitivity of the quasiparticle spectrum to disorder and boundary conditions, and distinguish between extended and localized states. Level correlations in metal grains,^{1–4} mesoscopic wires,⁵ quantum dots,⁶ SNS junctions,⁷ and superconducting vortex cores^{8–11} have recently been a subject of much active study. Most simply, level correlations are quantified by the mean square deviation $\langle \delta N_E^2 \rangle$ of the number N_E of levels in an energy interval of width E :

$$\langle \delta N_E^2 \rangle \equiv \langle [N_E - \langle N_E \rangle]^2 \rangle,$$

where the angular brackets denote disorder average.

In disordered systems, the energy levels of localized states are uncorrelated, and $\langle \delta N_E^2 \rangle \sim \langle N_E \rangle$. By contrast, extended states are strongly correlated,^{2,3} which is expressed in $\langle \delta N_E^2 \rangle$ scaling as only a logarithm of $\langle N_E \rangle$:

$$\langle \delta N_E^2 \rangle \approx \frac{2C}{\pi^2} \ln \langle N_E \rangle \ll \langle N_E \rangle, \quad (1)$$

where C is the number of the quasiparticle diffusion modes.¹² Formula (1) not only measures the level correlations, but also shows that, in a disordered system, the level number variance is driven by diffusion modes. Formula (1) holds for E smaller than the Thouless energy $E_c \equiv D/L^2$, where D is the diffusion constant, and L is the system size. At these energies, the level correlations are remarkably universal, with the constant C being defined solely by the fundamental symmetries of the system. The low energy universality classes of disordered systems have been classified based on the symmetry arguments.^{13,14}

In this article, I study quasiparticle level correlations in a disordered grain of a d-wave superconductor in the presence of both the time reversal (T) and the spin rotation invariance (S). I show that, in a wide intermediate energy range, the level correlations have the universal form (1), yet are different from those in a grain of a metal or of an s-wave superconductor (also invariant under T and S) – as well as different from the level correlations in any of the previously charted^{13,14} low energy universality classes. The main result is encapsulated in

the number C of the diffusion modes. In a grain of normal metal (or an s-wave superconductor) with both the time reversal and the spin rotation symmetries present, $C = 4$, since both the quasiparticle charge and the three components of the quasiparticle spin are conserved and propagate diffusively. By contrast, I show that, in a d-wave superconducting grain, the charge diffusion mode disappears, which leads to $C = 3$, despite the very same set of fundamental symmetries. This is a novel universal type of level correlations.

The reason behind this result is that the impurity scattering leads to the exchange of charge between the quasiparticle subsystem and the condensate. However, this process is sensitive to the momentum anisotropy of the gap, and its rate vanishes for an ideally isotropic gap, as noticed long ago¹⁵ in the context of the branch imbalance relaxation in NS junctions. On the contrary, in a d-wave superconductor, such a charge relaxation occurs at a time scale of order the impurity scattering time. As a result, compared with a normal metal (or an s-wave superconductor), the quasiparticle charge diffusion mode is missing in a d-wave superconductor, which reduces the constant C from four to three. This reduction is a robust many-body effect of the anisotropic pairing symmetry.

The plan of the paper is as follows. First, I show that, in an s-wave superconducting grain, the level correlations are essentially the same as in the normal state. Then I show how the gap anisotropy eliminates quasiparticle charge diffusion in a d-wave superconductor, and outline the corresponding microscopic calculation. Finally, I describe the applicability range of the result and its relation to the previous findings, and discuss the possible further developments.

S-wave superconductor. Consider an s-wave superconducting grain. In the approximation of a spatially uniform gap, superconductivity can be described as pairing of the exact time reversed eigenstates of the underlying metal.¹⁶ Thus, the exact quasiparticle energies E_n in the superconducting state can be expressed via the exact quasiparticle energies ε_n in the normal state as per $E_n = \sqrt{\varepsilon_n^2 + \Delta^2}$. Therefore, the exact density of states $\nu_S(E) \equiv \sum_n \delta[E - \sqrt{\varepsilon_n^2 + \Delta^2}]$ in the superconducting state is simply related to the exact density of

states $\nu_N(\varepsilon) \equiv \sum_n \delta[\varepsilon - \varepsilon_n]$ in the normal state:

$$\begin{aligned}\nu_S(E) &= \frac{E}{\sqrt{E^2 - \Delta^2}} \sum_n \delta[\sqrt{E^2 - \Delta^2} - \varepsilon_n] \\ &= \frac{E}{\sqrt{E^2 - \Delta^2}} \nu_N(\sqrt{E^2 - \Delta^2}).\end{aligned}$$

As a result, the density of states (DoS) dimensionless autocorrelation function $R_2^S(E, E') \equiv \frac{\langle \nu_S(E)\nu_S(E') \rangle}{\langle \nu_S(E) \rangle \langle \nu_S(E') \rangle} - 1$ in the superconducting state can be expressed through the DoS autocorrelation function in the normal state $R_2^N(\varepsilon, \varepsilon') \equiv \frac{\langle \nu_N(\varepsilon)\nu_N(\varepsilon') \rangle}{\langle \nu_N(\varepsilon) \rangle \langle \nu_N(\varepsilon') \rangle} - 1$ in the form

$$R_2^S(E, E') = R_2^N(\sqrt{E^2 - \Delta^2}, \sqrt{(E')^2 - \Delta^2}).$$

From this simple argument, it follows that the level correlations in an s-wave superconductor are identical (up to the change of variables) to those in the underlying normal metal and that, therefore, the diffusion modes in the two systems are the same.

Before moving further, it is instructive to classify the quasiparticle diffusion modes more precisely. As a two-particle process, diffusion amounts to a coherent propagation of a particle and a hole, carrying spin- $\frac{1}{2}$ each. These two spins can add to form a singlet, which corresponds to the charge degree of freedom – or a triplet, which corresponds to the three spin degrees of freedom. Both in s- and in d-wave superconductor, the condensate carries no spin, and thus a quasiparticle cannot exchange spin with the condensate. As a result, in a disordered spin-singlet superconductor, the quasiparticle spin does propagate diffusively, as it was recently re-emphasized.¹⁷ By contrast, the situation with the quasiparticle charge is more delicate. Perhaps the simplest way to observe this difference is by inspecting the Bogolyubov quasiparticle creation operator:

$$\gamma_{p\uparrow}^+ = u_p c_{p\uparrow}^+ + v_p c_{-p\downarrow}, \quad u_p^2 = \frac{1}{2} [1 - \frac{\varepsilon_p}{\sqrt{\varepsilon_p^2 + \Delta_p^2}}], \quad u_p^2 + v_p^2 = 1.$$

Impurity scattering is elastic, i.e. it conserves the quasiparticle energy $E_p = \sqrt{\varepsilon_p^2 + \Delta_p^2}$. In an s-wave superconductor with the uniform gap, Δ_p is a constant and, in the absence of the Andreev scattering that turns ε_p into $-\varepsilon_p$, the energy conservation implies conservation of u_p and v_p . This means conservation of the particle-hole content of a quasiparticle and leads to the effective charge conservation – even though a Bogolyubov quasiparticle, being a superposition of a particle and a hole, does not have a well defined charge quantum number. The same conclusion can be reached by considering directly the expectation value of the quasiparticle charge Q_p :

$$Q_p = u_p^2(+1) + v_p^2(-1) = \frac{\varepsilon_p}{\sqrt{\varepsilon_p^2 + \Delta_p^2}}.$$

In an s-wave superconductor, and in the absence of the Andreev processes, Q_p is conserved by the impurity scattering, which leads to the charge diffusion pole.

D-wave superconductor. By contrast, in a d-wave superconductor, the gap Δ_p has strong momentum dependence. Hence, even in the absence of the Andreev scattering processes, Q_p is not conserved by impurity scattering. In other words, in a d-wave superconductor, elastic scattering does not conserve the moduli of the Bogolyubov factors u_p and v_p , thus changing the particle-hole content of a quasiparticle. Physically, this means that the impurity scattering leads to the exchange of charge between the quasiparticle subsystem and the condensate at the time scale of order the impurity scattering time – and thus to the absence of the charge diffusion pole.

The same conclusion can be reached in a different and more formal way, by using the Ward identities, which are a consequence of the symmetries of the system.¹⁸ In the Nambu notations, the BCS Hamiltonian of a d-wave superconductor reads

$$\begin{aligned}H &= \int dr \Psi^\dagger \left[\tau_3 \varepsilon(\vec{p} - \frac{e}{c} \vec{A} \tau_3) + \tau_3 e \phi - \mu B + \tau_3 u \right] \Psi + \\ &\quad + \int dR dr \Psi^\dagger (R + \frac{r}{2}) \tau_1 \Delta(R, r) \Psi(R - \frac{r}{2}).\end{aligned}$$

Here $\Psi^\dagger \equiv (\psi_\uparrow^\dagger, \psi_\downarrow)$ is the Nambu spinor, τ_i are the Nambu matrices, R denotes the center of mass coordinate of a Cooper pair and r denotes the relative coordinate. The external fields are the vector potential \vec{A} , the electric potential ϕ , the Zeeman field B in the \hat{z} direction, and u is the impurity potential. The pair field $\Delta(R, r)$ has been chosen real for the sake of simplicity, and assumed to have the d-wave angular dependence.

The Hamiltonian above respects the gauge symmetry $\Psi \rightarrow U_g \Psi = \exp[i\tau_3 \frac{e}{\hbar c} \chi_g] \Psi$ (to be accompanied by the appropriate change of the potentials and of the gap function), and the \hat{z} axis spin rotations symmetry $\Psi \rightarrow U_s \Psi = \exp[i\frac{e}{\hbar c} \chi_s] \Psi$ (to be accompanied by the change of B).¹⁹ The sought identities can be obtained in the usual way,²⁰ by identifying the variation of the Green function under an infinitesimal transformation $G \rightarrow UGU^+$ with the first-order perturbative correction. In the absence of the external fields \vec{A} , B and ϕ , and assuming spatially uniform gap ($\Delta(R, r) = \Delta(r)$), the two symmetries imply two ($Q = 0, \Omega \equiv \varepsilon - \varepsilon' \rightarrow 0$) Ward identities for the disorder-averaged Green functions $G_{R,A}^{-1}(\varepsilon, p) = \varepsilon - \Sigma_{R,A}(\varepsilon) - \tau_1 \Delta_p - \tau_3 \varepsilon_p$, $\Sigma_R(\varepsilon) = \Sigma_A^*(\varepsilon)$ and the vertex renormalizations $\langle \tau_0 \rangle_{RA}$, $\langle \tau_3 \rangle_{RA}$ and $\langle \Delta_p \tau_2 \rangle_{RA}$:

$$-2i\tau_3 \Sigma_R''(\varepsilon) = (\varepsilon - \varepsilon') \langle \tau_3 \rangle_{RA} + 2i \langle \Delta_p \tau_2 \rangle_{RA}, \quad (2)$$

$$-2i\Sigma_R''(\varepsilon) = (\varepsilon - \varepsilon') \langle \tau_0 \rangle_{RA}. \quad (3)$$

The Nambu matrix τ_0 corresponds to the \hat{z} -component of the quasiparticle spin, and the vertex renormalization $\langle \tau_0 \rangle_{RA}$ describes its propagation. Similarly, τ_3 is

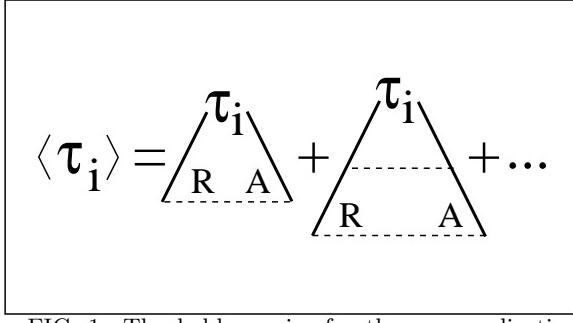


FIG. 1. The ladder series for the renormalization of the vertices $\langle \tau_i \rangle$.

the quasiparticle charge operator, whose propagation is accounted for by $\langle \tau_3 \rangle_{RA}$. The subscript RA means that the vertex joins a retarded and an advanced Green function at energies ε and ε' .²¹ For the illustration purposes, the leading approximation for the vertex corrections (the ladder series) is shown in Fig. 1.

Direct inspection of the ladder series shows that $\langle \Delta_p \tau_2 \rangle_{RA} \propto \langle \tau_3 \rangle_{RA}$, which leads to the conclusion that the vertex $\langle \tau_3 \rangle_{RA}$ remains finite as $(\varepsilon - \varepsilon') \rightarrow 0$. Thus, the quasiparticle charge does not propagate diffusively. At the same time, as seen from Eq. 3, the vertex $\langle \tau_0 \rangle_{RA}$, corresponding to the z -component of the spin, acquires a diffusion pole:

$$\langle \tau_0 \rangle_{RA} = \frac{2\Sigma''_R(\varepsilon)}{i(\varepsilon - \varepsilon')}.$$

Notice that, in a metal, the charge diffusion mode reappears, as seen by sending Δ_p to zero in Eq. (2).

The calculation. The microscopic calculation of the level correlations amounts to finding the DoS autocorrelation function $K(\varepsilon, \varepsilon') \equiv \langle \nu(\varepsilon)\nu(\varepsilon') \rangle - \langle \nu(\varepsilon) \rangle \langle \nu(\varepsilon') \rangle$. The mean square deviation of the number of levels in an energy interval of width E is then given simply by

$$\langle \delta N_E^2 \rangle = \int_E d\varepsilon d\varepsilon' K(\varepsilon, \varepsilon').$$

The calculation of $K(\varepsilon, \varepsilon')$ amounts to evaluating the Feynman diagram on Fig. 2, as done in³, and has to be performed in the four-spinor Balian-Werthamer space. The technical difference with regards to a metal being that now the Green functions reside in the matrix space, whereas the impurity ladder resides in the space of direct products of the two matrices. The calculation for nodal quasiparticles in a d-wave superconductor leads to Eq. 1, with $C = 3$, and its details will be published elsewhere.²²

The validity range. The applicability range of this calculation is set by the possibility to treat the gap as spatially uniform.²³ Hence the energy interval E should be much wider than the fluctuations of the gap. The latter scale is set by the perturbation of the gap due to the impurity potential. Neglecting the Coulomb interaction, the perturbation of the gap due to a single impurity has the form²⁴

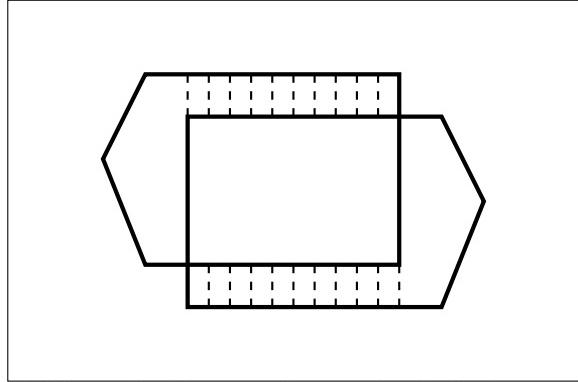


FIG. 2. The Feynman diagram for the leading term in the DoS autocorrelation function.

$$\delta\Delta(R) \sim \Delta [uN(0)] [\lambda N(0)] F(R),$$

where Δ is the value of the gap far from the impurity, $[uN(0)]$ is the dimensionless strength of the impurity potential, $[\lambda N(0)]$ is the dimensionless BCS coupling constant, and $F(R)$ is a function decaying to zero at the lengthscale of order the coherence length. Scattering off $\delta\Delta(R)$ is the Andreev reflection off the inhomogeneities of the gap, with the rate $\tau_A^{-1} \sim \tau^{-1} [\Delta/\epsilon_F]^2 [\lambda N(0)]^2 \ll \tau^{-1}$, where τ^{-1} is the impurity scattering rate. Thus, the main result of this paper holds for E much greater than τ_A^{-1} , and for the levels separated from the Fermi energy by more than τ_A^{-1} . At the same time, E must be much smaller than the Thouless energy $E_c = D/L^2$,³ which requires $\tau_A^{-1} \ll D/L^2$. The latter inequality bounds the grain size by

$$L \ll \frac{k_F l \xi}{[\lambda N(0)]},$$

which is much greater than the coherence length. Note that, for nodal quasiparticles in a d-wave superconductor, both l and ξ are functions of the quasiparticle energy ε and scale as $l(\varepsilon) \sim l\Delta/\varepsilon$, which further increases the upper bound on the grain size.

Before closing, it is instructive to put the main result of this work, Eq. (1) with $C = 3$, in context. Equation (1) (with different values of C) is commonly associated with the random matrix theory (RMT),¹³ which furnishes very general and powerful phenomenological framework for treating random systems, and allows a symmetry classification^{13,14} of possible universality classes. However, a crucial underlying assumption of the RMT is that *all* the matrix elements of the Hamiltonian (including the matrix elements of the gap) be random and drawn independently from a broad distribution. This requirement automatically rules out the possibility to distinguish superconductors of different pairing symmetry.

By contrast, the present work studies the intermediate energy limit when the matrix elements of the gap may be treated as completely non-random, the randomness being restricted to the diagonal of the Bogolyubov-de

Gennes Hamiltonian. In this limit, the Hamiltonian is only a “partly random” matrix, with the matrix elements of the gap fixed by the pairing symmetry. As shown above, the level correlations in such a “partly random” ensemble are sensitive to the momentum anisotropy of the pairing and are qualitatively different e.g. for s- and d-wave superconducting grains.

More importantly, this work shows that, in anisotropic superconductors, the condensate assumes the role of a “charge reservoir” coupled to the quasiparticle subsystem, and affects the level correlations. It would be interesting to study this problem further by explicitly including the dynamics of the condensate, especially in view of the cuprate superconductors as an obvious experimental object.

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- ²² Note that the statement that the level number variance $\langle \delta N_E^2 \rangle$ in a d-wave grain is 3/4 of that in an s-wave grain, holds also in the diffusive limit $E > E_c$, where $\langle \delta N_E^2 \rangle$ depends on the sample dimensionality and is a function of E/E_c .
- ²³ However, I do not see how the spatial variation of the gap, which has to be taken into account at lower energies, would revive the charge diffusion mode. There is a possibility that the main result, Eq. (1) with $C = 3$, is valid even beyond the formal applicability range of the calculation presented in this article.
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